may violate all notion of logical precision. An annoying situation of a type less vicious than the cases enumerated by Professor Lovitt is brought about by a text-book which defines the principal value of an inverse function as the smallest positive value.

In the last discussion Professor Rees shows how the motion of a body acted on only by the force of gravity and a resistance proportional to velocity may be readily and easily studied by the use of differential equations and initial conditions in vector form. This paper affords a good instance of the economy, both in notation and actual work, resulting from the use of single vector relations in place of triplets of scalar relations.

I. Note on the Quadrature of the Parabola.

By Otto Dunkel, Washington University.

The article by Professor Moritz entitled "On the Quadrature of the Parabola" in the November issue of the Monthly has suggested to the writer to present another derivation of the same result, since, in addition to being fairly simple, this second method follows directly the classic process of defining an area as the common limit of an inferior and a superior sum. This development might be found easy enough to serve as an illustrative example in the presentation of the summation formula in the integral calculus.

Let the equation of the curve be $y = x^m$, $m =$ a positive integer, and suppose that it is desired to obtain the area between the curve, the x-axis and the ordinates at $x = a$ and $x = b$, where b is greater than a and both are positive. Divide the interval from a to b on the x-axis into n subintervals, equal or unequal, and upon them as bases erect two sets of rectangles, the one inscribed and the other circumscribed. The sums of the areas of these rectangles are respectively,

$$I_n = \Sigma x_i^m(x_{i+1} - x_i), \qquad S_n = \Sigma x_{i+1}^m(x_{i+1} - x_i). \tag{1}$$

It will be shown that as n becomes infinite so that the length of the longest subinterval, δ, approaches zero, each of these sums approaches the same limit, which by the usual definition is the area desired. This limit will also be determined in the process.

It may easily be seen from a figure, especially when all the subintervals are equal, that

$$\text{Limit } (S_n - I_n) = 0. \tag{2}$$

This also follows algebraically, for

$$S_n - I_n = \Sigma(x_{i+1}^m - x_i^m)(x_{i+1} - x_i) \leqq \delta\Sigma(x_{i+1}^m - x_i^m).$$

In the latter summation all the terms cancel except the first and last, and hence the difference, $S_n - I_n$, is less than or equal to $\delta(b^m - a^m)$. Thus it follows that (2) is true.

A quantity will now be determined which lies between I_n and S_n and is independent of n. It is clear that x_{i+1}^m, $x_{i+1}^{m-1}x_i$, $x_{i+1}^{m-2}x_i^2$, $\cdots$, x_i^m form a decreasing sequence of $m + 1$ positive terms and hence their arithmetic mean is greater than the smallest term x_i^m. Using this inequality it follows that

$$I_n < \Sigma(x_{i+1} - x_i)\,\frac{x_{i+1}^m + x_{i+1}^{m-1}x_i + x_{i+1}^{m-2}x_i^2 + \cdots + x_i^m}{m+1},$$

$$< \frac{1}{m+1}\,\Sigma(x_{i+1}^{m+1} - x_i^{m+1}) = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}.$$

In a similar manner an inequality is found for S_n, and hence

$$0 < I_n < \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1} < S_n. \tag{3}$$

By representing the three quantities I_n, $b^{m+1}/(m + 1) - a^{m+1}/(m + 1)$, S_n by points on a straight line and by considering the meaning of (2) and (3) as applied to these points, it will be obvious that the common limit of I_n and S_n is $b^{m+1}/(m + 1) - a^{m+1}/(m + 1)$. This then is the expression for the desired area. If the equation of the curve is $y = px^m$, it will be readily seen that the above result must be multiplied by p.

The same method, with a slight amount of extra manipulation, may be used for negative values of m and also for fractional values, excepting, however, the special case $m = -1$.[1]

An elementary evaluation of the area of any segment of an ordinary parabola by means of special properties of the curve is given in the *Traité de Géométrie* by Rouché et Comberousse, 2d vol., p. 348 (8th ed., 1912). The properties here used are such as might be given in the ordinary text on analytics. A somewhat similar treatment occurs in the first volume of Goursat-Hedrick's *Mathematical Analysis*, p. 134, and is referred to as one of the processes used by Archimedes. Here the summation of a geometric series is employed, but this may be avoided and the proof simplified by comparing the areas of the interior triangles with certain corresponding exterior triangles. These two proofs are somewhat similar to the one employed by Professor Moritz.

II. Inverse Trigonometric Functions.

By W. V. Lovitt, Colorado College.

As I look over the available text-books on trigonometry the feeling grows upon me that they are hastily written and some topics inadequately treated. It is certain that many errors are present.

I have examined twenty-four different texts with special reference to their treatment of the inverse trigonometric functions. In five the treatment was so

[1] An elementary treatment of this case was given by the writer under the title, A Geometric Treatment of the Exponential Function, in *Washington University Studies*, scientific series, vol. 6, no. 2, p. 33.